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Interpolation theorems for variable exponent Lebesgue spaces

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Abstract

When Hardy–Littlewood maximal operator is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$ space we prove $[L^{p(\cdot)}(\mathbb{R}^n), BMO(\mathbb{R}^n)]_\theta = L^{q(\cdot)}(\mathbb{R}^n)$ where $q(\cdot) = p(\cdot)/(1-\theta)$ and $[L^{p(\cdot)}(\mathbb{R}^n), H^1(\mathbb{R}^n)]_\theta = L^{q(\cdot)}(\mathbb{R}^n)$ where $1/q(\cdot) = \theta + (1-\theta)/p(\cdot)$.

Keywords: Hardy space; Variable exponent Lebesgue space; Complex interpolation method; Calderon product

1. Introduction

The Lebesgue spaces $L^{p(\cdot)}(\mathbb{R}^n)$ with variable exponent and the corresponding variable Sobolev spaces $W^{k,p(\cdot)}(\mathbb{R}^n)$ are of interest for their applications to modelling problems in physics, and to the study of variational integrals and partial differential equations with non-standard growth condition (see [21]).

Given a measurable function $p(\cdot) : \mathbb{R}^n \to [1, \infty)$, $L^{p(\cdot)}(\mathbb{R}^n)$ denotes the set of measurable functions f on $\mathbb{R}^n$ such that for some $\lambda > 0$

$$\int_{\mathbb{R}^n} \left(\frac{|f(x)|}{\lambda}\right)^{p(x)} dx < \infty.$$

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This set becomes a Banach function space when equipped with the norm

$$\|f\|_{p(\cdot)} = \inf\left\{\lambda > 0\colon \int \left(\frac{|f(x)|}{\lambda}\right)^{p(x)} dx \leqslant 1\right\}.$$

There is interesting whether it is possible to transfer complex and real interpolation results to variable exponent Lebesgue spaces. It was shown in [19] that Riesz–Thorin theorem is valid on $L^{p(\cdot)}(\Omega)$ spaces, i.e., a linear operator T which is bounded from $L^{p_j(\cdot)}(\Omega)$ to $L^{p_j(\cdot)}(\Omega)$, $j = 0, 1$, is also bounded from $L^{p_\theta(\cdot)}(\Omega)$ to $L^{p_\theta(\cdot)}(\Omega)$. Here $p_\theta(\cdot)$ is defined usual way, $1/p_\theta(\cdot) = (1-\theta)/p_0(\cdot) + \theta/p_1(\cdot)$. In [7] was shown the stronger result $[L^{p_0(\cdot)}(\Omega), L^{p_1(\cdot)}(\Omega)]_\theta \approx L^{p_\theta(\cdot)}(\Omega)$, $1 < (p_j)_- \leqslant (p_j)_+ < \infty$ (below we will denote $p_- = \operatorname{ess\,inf} p(\cdot)$ and $p_+ = \operatorname{ess\,sup} p(\cdot)$), where $[A_1, A_2]_\theta$ denotes the complex interpolation space of the Banach spaces A_1 and A_2 (see definition in [1]).

Recall that $BMO(\mathbb{R}^n)$ denotes the space of functions of bounded mean oscillation with the seminorm

$$\|f\|_{BMO} = \sup \frac{1}{|Q|} \int\limits_Q \left|f(y) - f_Q\right| dy,$$

where $f_Q = \frac{1}{|Q|}\int_Q f(x)\,dx$ and supremum is taken over all cubes in $\mathbb{R}^n$.

Among the various equivalent characterizations of the Hardy space $H^1(\mathbb{R}^n)$ one of the simplest is using a maximal function (see [22]): fix a function $\phi \in S(\mathbb{R}^n)$ with $\int \phi(x)\,dx = 1$ and define the maximal function of a distribution $f \in S'(\mathbb{R}^n)$ by

$$M_\phi f(x) = \sup_{t>0}\left|f * \phi_t(x)\right|, \quad x \in \mathbb{R}^n.$$

Then $f \in H^1(\mathbb{R}^n)$ if $M_\phi f \in L^1(\mathbb{R}^n)$, and one can set

$$\|f\|_{H^1} = \|M_\phi f\|_1.$$

In this paper we study complex interpolation spaces $[L^{p(\cdot)}(\mathbb{R}^n), BMO(\mathbb{R}^n)]_\theta$ and $[L^{p(\cdot)}(\mathbb{R}^n), H^1(\mathbb{R}^n)]_\theta$ when Hardy–Littlewood maximal operator is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$. We follow the approach of Frazier and Jawerth [9] which takes advantage of the fact that, unlike distribution spaces $\dot{F}^\alpha_{p,q}$, the sequence spaces $\dot{f}^\alpha_{p,q}$ are quasi-Banach lattices. Hence, by computing Calderon products of $\dot{f}^\alpha_{p,q}$ spaces we can deduce complex interpolation results for $\dot{F}^\alpha_{p,q}$ spaces.

Let $B(x,r)$ denote the open ball in $\mathbb{R}^n$ of radius r and center x. By $|B(x,r)|$ we denote n-dimensional Lebesgue measure of $B(x,r)$. The Hardy–Littlewood maximal operator M is defined on locally integrable function f on $\mathbb{R}^n$ by the formula

$$Mf(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int\limits_{B(x,r)} \left|f(y)\right| dy.$$

By $\mathcal{P}(\mathbb{R}^n)$ denote the class of all exponents $p(\cdot)$ with property $1 < a \leqslant p(t) \leqslant b < \infty$; $t \in \mathbb{R}^n$ and define $\mathcal{B}(\mathbb{R}^n)$ to be the set of exponents $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ such that M is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$. Below by $p'(\cdot)$ we denote the conjugate exponent of $p(\cdot)$ $(1/p(t) + 1/p'(t) = 1,\ t \in \mathbb{R}^n)$.

In harmonic analysis a fundamental operator is the Hardy–Littlewood maximal operator M. In many applications a crucial step has been to show that operator M is bounded on a variable L^p space. Note that many classical operators in harmonic analysis such as singular integrals, commutators and fractional integrals are bounded on the variable Lebesgue space $L^{p(\cdot)}$ whenever the Hardy–Littlewood maximal operator is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$. Conditions for the boundedness of the maximal and singular operators on spaces $L^{p(\cdot)}(\mathbb{R}^n)$ have been studied in [6,5,20,3,13,14,4,7].

We will prove the following

Theorem 1.1. *Let $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$ and $0 < \theta < 1$. Then*

$$\big[L^{p(\cdot)}\big(\mathbb{R}^n\big), BMO\big(\mathbb{R}^n\big)\big]_\theta = L^{q(\cdot)}\big(\mathbb{R}^n\big), \quad \textit{where } q(\cdot) = \frac{p(\cdot)}{1-\theta}, \tag{1.1}$$

$$\big[L^{p(\cdot)}\big(\mathbb{R}^n\big), H^1\big(\mathbb{R}^n\big)\big]_\theta = L^{q(\cdot)}\big(\mathbb{R}^n\big), \quad \textit{where } 1/q(\cdot) = \theta + (1-\theta)/p(\cdot). \tag{1.2}$$

For the case $p(\cdot) = \text{const}$, this is the classical result of Fefferman and Stein [8]. Later several authors have extended Fefferman–Stein's complex interpolation result in the sense of replacing L^p on left side by L^∞ or BMO (see [11,18,24] for different approaches to this result).

As a consequence we obtain

Corollary 1.2. *Let $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$, $0 < \theta < 1$ and $q(\cdot) = \frac{p(\cdot)}{1-\theta}$. Then there exists a constant C such that*

$$\|f\|_{q(\cdot)} \leqslant C\|f\|_{p(\cdot)}^{1-\theta}\|f\|_{BMO}^{\theta} \tag{1.3}$$

holds for all $f \in L^{p(\cdot)}(\mathbb{R}^n) \cap BMO(\mathbb{R}^n)$.

Note that in case $p(\cdot) \equiv \text{const}$ (1.3) was shown by Chen and Zhu [2] by using the John–Nirenberg type inequality (see also [16]).

2. Characterization of variable exponent Lebesgue space

Many function spaces arising in harmonic analysis admit decompositions into simpler building blocks, often called atoms or molecules, that have some additional desirable properties.

One of the possible directions, where decomposition techniques are very useful, is the study of a large class of general Triebel–Lizorkin spaces $\dot{F}^\alpha_{p,q}$ (homogeneous) and $F^\alpha_{p,q}$ (inhomogeneous), $\alpha \in \mathbb{R}$, $0 < p, q \leqslant \infty$ which includes many well-known classical function spaces. In particular, $L^p \approx \dot{F}^0_{p,2} \approx F^0_{p,2}$ when $1 < p < \infty$ (here $\approx$ means that the (quasi)-norms are equivalent) and $\dot{F}^0_{p,2} \approx H^p$ when $0 < p \leqslant 1$. The atomic and molecular decomposition results for homogeneous Triebel–Lizorkin spaces were first obtained by Frazier and Jawerth [9].

For $\nu \in \mathbb{Z}$ and $k \in \mathbb{Z}^n$, let $Q_{\nu k}$ be the dyadic cube $2^{-\nu}[0,1]^n + k$. Let $\mathcal{D}$ be the collection of dyadic cubes in $\mathbb{R}^n$. For a cube Q let x_Q denote the "lower left corner".

The set $\mathcal{S}(\mathbb{R}^n)$ denotes the usual Schwartz space of rapidly decreasing complex-valued functions and $\mathcal{S}'(\mathbb{R}^n)$ denotes the dual space of tempered distributions. We denote the Fourier transform of φ by $\widehat{\varphi}$. Let functions $\varphi, \psi \in \mathcal{S}(\mathbb{R}^n)$ satisfy the following conditions:

$$\operatorname{supp}\widehat{\varphi}, \widehat{\psi} \subset \left\{\xi \in \mathbb{R}^n\colon \frac{1}{2} \leqslant |\xi| \leqslant 2\right\}, \tag{2.1}$$

$$\text{and} \quad \left|\widehat{\varphi}(\xi)\right|, \left|\widehat{\varphi}(\xi)\right| \geqslant c > 0 \quad \text{when } \frac{3}{5} \leqslant |\xi| \leqslant \frac{5}{3}, \tag{2.2}$$

$$\sum_{\nu\in\mathbb{Z}} \widehat{\widetilde{\varphi}}\big(2^{-\nu}\xi\big)\widehat{\psi}\big(2^{-\nu}\xi\big) = 1 \quad \text{if } \xi \neq 0. \tag{2.3}$$

Here $\widetilde{\varphi}(x) = \overline{\varphi(-x)}$. We set $\varphi_\nu(x) = 2^{\nu n}\varphi(2^\nu x)$ and $\psi_\nu(x) = 2^{\nu n}\psi(2^\nu x)$, $\nu \in \mathbb{Z}$. Define $\varphi_Q(x) = |Q|^{1/2}\varphi_\nu(x - x_Q)$ if $Q = Q_{\nu k}$, and similarly define ψ_Q.

For φ and ψ satisfying (2.1)–(2.3), the φ-transform S_φ is the map taking each $f \in \mathcal{S}'(\mathbb{R}^n)/\mathcal{P}$ (the space of tempered distributions modulo polynomials) to the sequence $S_\varphi f = \{(S_\varphi f)_Q\}_{Q\in\mathcal{D}}$ defined by $(S_\varphi f)_Q = \langle f, \varphi_Q\rangle$ for Q dyadic. Here $\langle\cdot,\cdot\rangle$ denotes the usual inner product on $L^2(\mathbb{R}^n;\mathbb{C})$. The inverse of φ-transform T_ψ is the map taking a sequence $s = \{s_Q\}_{Q\in\mathcal{D}}$ to $T_\psi s = \sum_{Q\in\mathcal{D}} s_Q\psi_Q$.

The starting point in the theory of discrete φ-transforms of Frazier and Jawerth [9] is the representation formula for tempered distributions: If $f \in \mathcal{S}'(\mathbb{R}^n)/\mathcal{P}$, then

$$f(x) = \sum_{Q\in\mathcal{Q}} \langle f, \varphi_Q\rangle\psi_Q(x),$$

where the convergence of the above series, as well as the equality, is in $\mathcal{S}'(\mathbb{R}^n)/\mathcal{P}$.

Motivated by the classical definition of homogeneous Triebel–Lizorkin spaces by Triebel [23], Frazier, Jawerth and Weiss [9,10] we define variable discrete homogeneous Triebel–Lizorkin spaces $\dot{f}^0_{p(\cdot),2}$ as follows: for $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ let $\dot{f}^0_{p(\cdot),2}$ be the collection of all complex-valued sequences $s = \{s_Q\}_{Q\in\mathcal{D}}$ such that

$$\|s\|_{\dot{f}^0_{p(\cdot),2}} = \left\|\left(\sum_{Q\in\mathcal{D}} \big(|s_Q|\widetilde{\chi}_Q\big)^2\right)^{1/2}\right\|_{p(\cdot)} < \infty,$$

where $\widetilde{\chi}_Q = |Q|^{-1/2}\chi_Q$ is the L^2-normalized characteristic function of Q.

Theorem 2.1. *Suppose $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$. The operators $S_\varphi : L^{p(\cdot)}(\mathbb{R}^n) \to \dot{f}^0_{p(\cdot),2}$ and $T_\psi : \dot{f}^0_{p(\cdot),2} \to L^{p(\cdot)}(\mathbb{R}^n)$ are bounded. Furthermore, $T_\psi \circ S_\varphi$ is the identity on $L^{p(\cdot)}(\mathbb{R}^n)$. In particular $\|f\|_{p(\cdot)} \approx \|S_\varphi f\|_{\dot{f}^0_{p(\cdot),2}}$.*

The above norm estimate is consequence of the extrapolation theorem given by Cruz-Uribe, Fiorenza, Martell and Perez [4] and the weighted norm inequalities for

$$\mathcal{W}f(x) = \left(\sum_{Q\in\mathcal{D}} \big(\big|\langle f, \varphi_Q\rangle\big|\widetilde{\chi}_Q\big)^2\right)^{1/2}$$

function, given by Frazier and Jawerth [9, Proposition 10.14]. We describe these results.

Under a weight we mean a non-negative, locally integrable function w. When $1 < p < \infty$, we say $w \in A_p$ if for every cube Q

$$\frac{1}{|Q|}\int_Q w(x)\,dx\left(\frac{1}{|Q|}\int_Q w(x)^{1-p'}\,dx\right)^{p-1} \leqslant C < \infty.$$

By $A_{p,w}$ we denote the infimum over the constants on the right-hand side of the last inequality. By $\mathcal{F}$ we will denote a family of ordered pairs of non-negative, measurable functions (f,g). We say that an inequality

$$\int_{\mathbb{R}^n} f(x)^{p_0} w(x)\,dx \leqslant C\int_{\mathbb{R}^n} g(x)^{p_0} w(x)\,dx \quad (0 < p_0 < \infty) \tag{2.4}$$

holds for any $(f,g) \in \mathcal{F}$ and $w \in A_q$ (for some q, $1 < q < \infty$) if it holds for any pair in $\mathcal{F}$ such that the left-hand side is finite, and the constant C depends only on p_0 and on the constant $A_{q,w}$.

Theorem 2.2. *(See [4].) Given a family $\mathcal{F}$, assume that* (2.4) *holds for some $1 < p_0 < \infty$, for every weight $w \in A_{p_0}$ and for all $(f,g) \in \mathcal{F}$. Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ be such that there exists $1 < p_1 < p_-$, with $(p(\cdot)/p_1)' \in \mathcal{B}(\mathbb{R}^n)$. Then*

$$\|f\|_{p(\cdot)} \leqslant C\|g\|_{p(\cdot)}$$

for all $(f,g) \in \mathcal{F}$ such that $f \in L^{p(\cdot)}(\mathbb{R}^n)$.

Let w be a non-negative function satisfying "doubling condition" $w(2Q) \leqslant Cw(Q)$, where, for a measurable set E $w(E) = \int_E w(x)\,dx$. The discrete Triebel–Lizorkin weighted sequence space $\dot{f}^0_{p,2}(w)$ is defined (see [9]) as the collection of all complex-valued sequences s such that

$$\|s\|_{\dot{f}^0_{p,2}(w)} = \left\|\left(\sum_{Q\in\mathcal{D}}\left(|s_Q|\widetilde{\chi}_Q\right)^2\right)^{1/2}\right\|_{L^p(w)} < \infty,$$

where $L^p(w)$ is weighted Lebesgue space.

Theorem 2.3. *(See [9].) Suppose $p \in (1,\infty)$, $w \in A_p$. The operators $S_\varphi : L^p(w) \to \dot{f}^0_{p,2}(w)$ and $T_\psi : \dot{f}^0_{p,2}(w) \to L^p(w)$ are bounded. Furthermore, $T_\psi \circ S_\varphi$ is the identity on $L^p(w)$. In particular*

$$\int_{\mathbb{R}^n} \left|f(x)\right|^p w(x)\,dx \approx \int_{\mathbb{R}^n} \mathcal{W}f(x)^p w(x)\,dx. \tag{2.5}$$

Proof of Theorem 2.1. From the assumption $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$ we get that there exists $1 < p_1 < p_-$ with $(p(\cdot)/p_1)' \in \mathcal{B}(\mathbb{R}^n)$ [6, Theorem 8.1]. An application of Theorem 2.2 and (2.5) with the pairs $(\mathcal{W}f, |f|)$ gives the norm estimate $\|S_\varphi f\|_{\dot{f}^0_{p(\cdot),2}} \leqslant C\|f\|_{p(\cdot)}$, provided $f \in C_0^\infty(\mathbb{R}^n)$. Note that $C_0^\infty(\mathbb{R}^n)$ is dense in $L^{p(\cdot)}(\mathbb{R}^n)$ and consequently this inequality is also valid for all $f \in L^{p(\cdot)}(\mathbb{R}^n)$. Analogously we obtain opposite direction inequality. $\square$

The extrapolation theorem given by Cruz-Uribe, Fiorenza, Martell and Perez, implies among other things the Fefferman–Stein vector-valued inequality for variable exponent Lebesgue spaces.

Theorem 2.4. *(See [4].) Let $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$ and $1 < q < \infty$. Then*

$$\big\| \|Mf_i\|_{l^q} \big\|_{p(\cdot)} \leqslant C \big\| \|f_i\|_{l^q} \big\|_{p(\cdot)}.$$

Theorem 2.5. *Let $\varepsilon > 0$, $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$. Suppose that for each dyadic cube Q, $E_Q \subseteq Q$ is a measurable set with $|E_Q|/|Q| \geqslant \varepsilon$. Then*

$$\big\|\{s_Q\}_Q\big\|_{\dot f^0_{p(\cdot),2}} \approx \left\| \left(\sum_{Q\in\mathcal{D}} \big(|s_Q|\widetilde{\chi}_{E_Q}\big)^2 \right)^{1/2} \right\|_{p(\cdot)}, \tag{2.6}$$

where $\widetilde{\chi}_{E_Q} = |E_Q|^{-1/2}\chi_{E_Q}$.

Proof. Since $\widetilde{\chi}_{E_Q} \leqslant \varepsilon^{-1/2}\widetilde{\chi}_Q$, one direction is trivial. For the other, note that for all $A > 0$, $\chi_Q \leqslant \varepsilon^{-1/A}(M(\chi^A_{E_Q}))^{1/A}$, where M denotes the Hardy–Littlewood operator. Select A such that $p_-/A, 2/A > 1$. Note that $\|f\|_{p(\cdot)} = \|f^A\|^{1/A}_{p(\cdot)/A}$ and by Theorem 2.4 we have

$$\begin{aligned}
\big\|\{s_Q\}_Q\big\|_{\dot f^0_{p(\cdot),2}} &\leqslant \varepsilon^{-1/A} \left\| \left(\sum_{Q\in\mathcal{D}} \big(M\big(|s_Q|\widetilde{\chi}_{E_Q}\big)^A\big)^{2/A} \right)^{A/2} \right\|^{1/A}_{p(\cdot)/A} \\
&\leqslant C\varepsilon^{-1/A} \left\| \left(\sum_{Q\in\mathcal{D}} \big(|s_Q|\widetilde{\chi}_{E_Q}\big)^2 \right)^{1/2} \right\|_{p(\cdot)}. \qquad \square
\end{aligned}$$

Corollary 2.6. *Let $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$. Then $s = \{s_Q\}_{Q\in\mathcal{D}} \in \dot f^0_{p(\cdot),2}$ if and only if for each Q dyadic there is a subset $E_Q \subset Q$ with $|E_Q|/|Q| > 1/2$ (or any other fixed number $0 < \varepsilon < 1$) such that*

$$\left\| \left(\sum_{Q\in\mathcal{D}} \big(|s_Q|\widetilde{\chi}_{E_Q}\big)^2 \right)^{1/2} \right\|_{p(\cdot)} < \infty.$$

For space $BMO(\mathbb{R}^n)$ we have the following identification $\|f\|_{BMO} \approx \|S_\varphi f\|_{\dot f^0_{\infty,2}}$ (see [10]), where discrete Triebel–Lizorkin sequence space $\dot f^0_{\infty,2}$ is defined as the collection of all complex-valued sequences $s = \{s_Q\}_{Q\in\mathcal{D}}$ such that

$$\|s\|_{\dot f^0_{\infty,2}} = \sup_{P\ \text{dyadic}} \frac{1}{|P|} \int_P \left(\sum_{Q\subset P} \big(|s_Q|\widetilde{\chi}_Q(x)\big)^2 \right)^{1/2} dx < \infty.$$

Note also that $s = \{s_Q\}_{Q\in\mathcal{D}} \in \dot{f}^0_{\infty,2}$ if and only if for each Q there is a subset $E_Q \subset Q$ with $|E_Q|/|Q| > 1/2$ (or any other fixed number $0 < \varepsilon < 1$) such that

$$\left\| \left(\sum_{Q\in\mathcal{D}} \left(|s_Q|\widetilde{\chi}_{E_Q}\right)^2 \right)^{1/2} \right\|_{L^\infty} < +\infty. \tag{2.7}$$

Moreover, the infimum of this expression over all such collections $\{E_Q\}_{Q\in\mathcal{D}}$ is equivalent to $\|s\|_{\dot{f}^0_{\infty,2}}$ [9, Corollary 5.6].

For Hardy space $H^1(\mathbb{R}^n)$ we have the following identification $\|f\|_{H^1} \approx \|S_\varphi f\|_{\dot{f}^0_{1,2}}$ (see [10]), where $\dot{f}^0_{1,2}$ is defined as the collection of all complex-valued sequences $s = \{s_Q\}_{Q\in\mathcal{D}}$ such that

$$\|s\|_{\dot{f}^0_{1,2}} = \int\limits_{\mathbb{R}^n} \left(\sum_{Q\in\mathcal{D}} \left(|s_Q|\widetilde{\chi}_Q(x)\right)^2 \right)^{1/2} dx < \infty.$$

3. The Calderon product and interpolation property

Let (Ω, μ) be a complete σ-finite measure space. By S we denote the collection of all real-valued measurable functions on Ω. A Banach subspace X in S is said to be Banach lattices on (Ω, μ) if:

(1) the norm $\|f\|_X$ is defined for every measurable function f and $f \in X$ if and only if $\|f\|_X < \infty$, $\|f\|_X = 0$ if and only if $f = 0$ a.e.;
(2) $\||f|\|_X = \|f\|_X$ for all $f \in X$;
(3) if $0 \leqslant f \leqslant g$ a.e., then $\|f\|_E \leqslant \|g\|_E$;
(4) if $0 \leqslant f_n \uparrow f$ a.e., then $\|f_n\|_X \uparrow \|f\|_X$ (Fatou property).

A Banach lattice modelled on the discrete set is called a Banach sequence lattice (below we assume that $\operatorname{supp} X = \Omega$).

If X is a Banach lattice on Ω, then the Köthe dual X' of X is a Banach lattice which can be identified with the space of all functionals possessing an integral representation. That is,

$$X' = \left\{ f \in S;\ \sup_{\|g\|_X < 1} \int\limits_{\Omega} |fg|\, d\mu < \infty \right\}.$$

The space X' is a Banach lattice on Ω and a closed normed subspace of conjugate space X^*.

Recall that X is order continuous if for any $f \in X$ and $f_n \leqslant f$ with $|f_n| \to 0$ a.e., $\|f_n\|_X \to 0$. Note that if X is order continuous then $E^* = E'$ and $E = (E^*)'$ (see [12]).

Suppose that X_0 and X_1 are Banach lattices on Ω. If $0 < \theta < 1$, the Calderon product $X_0^{1-\theta} X_1^\theta$ of X_0 and X_1 is defined to be the set of μ-measurable functions f on Ω such that there exist $v \in X_0$ with $\|v\|_{X_0} \leqslant 1$, $w \in X_1$, with $\|w\|_{X_1} \leqslant 1$, and $\lambda > 0$ such that

$$\left|f(x)\right| \leqslant \lambda \left|v(x)\right|^{1-\theta} \left|w(x)\right|^\theta \quad \text{for } \mu \text{ a.e. } x. \tag{3.1}$$

We set

$$\|u\|_{X_0^{1-\theta}X_1^\theta}=\inf\bigl\{\lambda>0\colon\ (3.1)\text{ holds with }\|v\|_{X_0}\leqslant 1\text{ and }\|w\|_{X_1}\leqslant 1\bigr\}.$$

In the sequel we will need the following obvious inequality: if $f\in X_0$ and $g\in X_1$ then

$$\bigl\|f^{1-\theta}g^\theta\bigr\|_{X_0^{1-\theta}X_1^\theta}\leqslant\|f\|_{X_0}^{1-\theta}\|g\|_{X_1}^\theta. \tag{3.2}$$

Indeed if $\|f\|_{X_0},\|g\|_{X_1}\neq 0$ then for $u=f/\|f\|_{X_0}$ and $v=g/\|g\|_{X_1}$ we have $\|u\|_{X_0}=\|v\|_{X_1}=1$ and $|f|^{1-\theta}|g|^\theta=\|f\|_{X_0}^{1-\theta}\|g\|_{X_1}^\theta|u|^{1-\theta}|v|^\theta$.

We will use further also the well-known fact that (see [1,17])

$$\bigl(X_0^{1-\theta}X_1^\theta\bigr)'=\bigl(X_0'\bigr)^{1-\theta}\bigl(X_1'\bigr)^\theta. \tag{3.3}$$

Theorem 3.1. *Let $p(\cdot)\in\mathcal{B}(\mathbb{R}^n)$, $0<\theta<1$ and $q(\cdot)=\frac{p(\cdot)}{1-\theta}$. Then*

$$\bigl(\dot f^0_{p(\cdot),2}\bigr)^{1-\theta}\bigl(\dot f^0_{\infty,2}\bigr)^\theta=\dot f^0_{q(\cdot),2}. \tag{3.4}$$

Proof. Let $X_0=\dot f^0_{p(\cdot),2}$ and $X_1=\dot f^0_{\infty,2}$. Suppose $s\in X_0^{1-\theta}X_1^\theta$ and $\|s\|_{X_0^{1-\theta}X_1^\theta}=1$. Then, there exist sequences $r=\{r_Q\}_{Q\in\mathcal{D}}$ and $t=\{t_Q\}_{Q\in\mathcal{D}}$ such that

$$\|r\|_{X_0}\leqslant 1,\qquad \|t\|_{X_1}\leqslant 1,\qquad |s_Q|\leqslant 2|r_Q|^{1-\theta}|t_Q|^\theta\quad\text{for all }Q\in\mathcal{D}.$$

Let $\varepsilon=3/4$, then there exist sets $E_Q\subset Q$ for each dyadic cube Q (see (2.6), (2.7)) such that $|E_Q|/|Q|>3/4$ and

$$\|r\|_{X_0}\approx\biggl\|\biggl(\sum_{Q\in\mathcal{D}}\bigl(|r_Q|\widetilde\chi_{E_Q}\bigr)^2\biggr)^{1/2}\biggr\|_{p(\cdot)}, \tag{3.5}$$

$$\|t\|_{X_1}\approx\biggl\|\biggl(\sum_{Q\in\mathcal{D}}\bigl(|t_Q|\widetilde\chi_{E_Q}\bigr)^2\biggr)^{1/2}\biggr\|_{L^\infty}. \tag{3.6}$$

Applying Hölder's inequality with conjugate exponents $1/(1-\theta)$ and $1/\theta$ we obtain

$$\biggl(\sum_{Q\in\mathcal{D}}\bigl(|s_Q|\widetilde\chi_{E_Q}\bigr)^2\biggr)^{1/2}\leqslant 2\biggl(\sum_{Q\in\mathcal{D}}\bigl(|r_Q|\widetilde\chi_{E_Q}\bigr)^2\biggr)^{(1-\theta)/2}\biggl(\sum_{Q\in\mathcal{D}}\bigl(|t_Q|\widetilde\chi_{E_Q}\bigr)^2\biggr)^{\theta/2}.$$

Eqs. (3.5) and (3.6) give

$$\int_{\mathbb{R}^n}\biggl(\sum_{Q\in\mathcal{D}}\bigl(|s_Q|\widetilde\chi_{E_Q}(x)\bigr)^2\biggr)^{q(x)/2}dx\leqslant C\int_{\mathbb{R}^n}\biggl(\sum_{Q\in\mathcal{D}}\bigl(|r_Q|\widetilde\chi_{E_Q}(x)\bigr)^2\biggr)^{(1-\theta)q(x)/2}dx\leqslant C.$$

According to Theorem 2.5 we have $\|s\|_{\dot{f}^0_{q(\cdot),2}} \leqslant C$ and consequently

$$\left(\dot{f}^0_{p(\cdot),2}\right)^{1-\theta}\left(\dot{f}^0_{\infty,2}\right)^{\theta} \subset \dot{f}^0_{q(\cdot),2}.$$

Suppose $s \in \dot{f}^0_{q(\cdot),2}$. For $k \in \mathbb{Z}$, define

$$\Omega_k = \left\{x \in \mathbb{R}^n\colon \left(\sum_{Q\in\mathcal{D}} \left(|s_Q|\widetilde{\chi}_Q(x)\right)^2\right)^{1/2} > 2^k\right\},$$
$$\mathcal{Q}_k = \left\{Q \in \mathcal{D}\colon |Q \cap \Omega_k| \geqslant |Q|/2 \text{ and } |Q \cap \Omega_{k+1}| < |Q|/2\right\}.$$

Note that if $Q \notin \bigcup_{k\in\mathbb{Z}} \mathcal{Q}_k$ then $s_Q = 0$ and in this case set $r_Q = t_Q = 0$. Otherwise, if $Q \in \mathcal{Q}_k$ for some $k \in \mathbb{Z}$, then we set

$$r_Q = |s_Q|/A_Q \quad \text{and} \quad t_Q = |s_Q|/B_Q$$

where $A_Q = 2^{k(1-1/(1-\theta))} = 2^{k\delta}$, $B_Q = 2^k$. A direct calculation shows that $|s_Q| = |r_Q|^{1-\theta}|t_Q|^{\theta}$. In addition, we claim that $\|r\|_{X_0}$, $\|t\|_{X_1} \leqslant C$.

To prove $\|r\|_{X_0} \leqslant C$ inequality we use Theorem 2.5 with $E_Q = Q \cap \Omega_k$, $Q \in \mathcal{Q}_k$,

$$\begin{aligned}
&\int\limits_{\mathbb{R}^n} \left(\sum_{Q\in\mathcal{D}} \left(|r_Q|\widetilde{\chi}_{E_Q}(x)\right)^2\right)^{p(x)/2} dx \\
&\quad = \int\limits_{\mathbb{R}^n} \left(\sum_{k\in\mathbb{Z}} \sum_{Q\in\mathcal{Q}_k} \left(2^{-k\delta}|s_Q|\widetilde{\chi}_{E_Q}(x)\right)^2\right)^{p(x)/2} dx \\
&\quad \leqslant \int\limits_{\mathbb{R}^n} \left(\sum_{k\in\mathbb{Z}} \chi_{\Omega_k} \sum_{Q\in\mathcal{Q}_k} \left(2^{-k\delta}|s_Q|\widetilde{\chi}_{E_Q}(x)\right)^2\right)^{p(x)/2} dx \\
&\quad \leqslant \int\limits_{\mathbb{R}^n} \left(\sum_{Q\in\mathcal{D}} \left(|s_Q|\widetilde{\chi}_Q(x)\right)^2\right)^{p(x)/2(1-\theta)} dx \\
&\quad = \int\limits_{\mathbb{R}^n} \left(\sum_{Q\in\mathcal{D}} \left(|s_Q|\widetilde{\chi}_Q(x)\right)^2\right)^{q(x)/2} dx \leqslant C.
\end{aligned}$$

Here, we used that $\delta < 0$ and

$$2^{-k\delta}\chi_{\Omega_k} \leqslant \left(\sum_{Q\in\mathcal{D}} \left(|s_Q|\widetilde{\chi}_Q\right)^2\right)^{-\delta/2}.$$

To prove $\|t\|_{X_1} \leqslant C$ inequality we use a similar argument as above by redefining $E_Q = Q \cap (\Omega_{k+1})^c$, $Q \in \mathcal{Q}_k$,

$$\|t\|_{X_1} \leqslant C\left\|\left(\sum_{k\in\mathbb{Z}}\sum_{Q\in\mathcal{Q}_k}\left(2^{-k}|s_Q|\widetilde{\chi}_{E_Q}\right)^2\right)^{1/2}\right\|_{L^\infty}$$

$$\leqslant C\left\|\left(\sum_{k\in\mathbb{Z}}\chi_{(\Omega_{k+1})^c}\sum_{Q\in\mathcal{Q}_k}\left(2^{-k}|s_Q|\widetilde{\chi}_{E_Q}\right)^2\right)^{1/2}\right\|_{L^\infty}$$

$$\leqslant C\left\|\left(\sum_{Q\in\mathcal{D}}\left(|s_Q|\widetilde{\chi}_{E_Q}\right)^2\right)^{1/2}\right\|_{L^\infty} \leqslant C.$$

Hence, we have $s \in X_0^{1-\theta}X_1^{\theta}$. $\square$

Proof of Theorem 1.1. We can obtain results regarding complex interpolation from Theorem 3.1. Let $[X_0, X_1]_\theta$ denote the space obtained from X_0 and X_1 by the complex interpolation method. Suppose X_0 and X_1 are Banach lattices on a measure space Ω, and let $X = X_0^{1-\theta}X_1^{\theta}$ for some $\theta \in (0,1)$. Calderon in [1] shows that $[X_0, X_1]_\theta = X_0^{1-\theta}X_1^{\theta}$ under the hypothesis that X has Fatou property. Note that $L^{p(\cdot)}$, if $p(\cdot) \in L^\infty$, satisfies the above property by the Lebesgue dominated convergence theorem (see [15]). Hence, we obtain

$$\left[\dot{f}^0_{p(\cdot),2}, \dot{f}^0_{\infty,2}\right]_\theta = \dot{f}^0_{q(\cdot),2}.$$

Applying Theorem 2.1 we obtain (1.1).

To prove (1.2), we first observe that $(L^{p(\cdot)}(\mathbb{R}^n))^* = (L^{p(\cdot)}(\mathbb{R}^n))' = L^{p'(\cdot)}(\mathbb{R}^n)$ (see [15]) and consequently $(\dot{f}^0_{p(\cdot),2})^* = (\dot{f}^0_{p(\cdot),2})' = \dot{f}^0_{p'(\cdot),2}$. Note also that $(\dot{f}^0_{1,2})^* = (\dot{f}^0_{1,2})' = \dot{f}^0_{\infty,2}$ and $\dot{f}^0_{1,2} = (\dot{f}^0_{\infty,2})'$ ($\dot{f}^0_{1,2}$ is order continuous Banach lattices).

According to (3.3) we can write

$$\left(\left(\dot{f}^0_{p'(\cdot),2}\right)^{1-\theta}\left(\dot{f}^0_{\infty,2}\right)^{\theta}\right)' = \left(\left(\dot{f}^0_{p'(\cdot),2}\right)'\right)^{1-\theta}\left(\left(\dot{f}^0_{\infty,2}\right)'\right)^{\theta}$$

$$= \left(\dot{f}^0_{p(\cdot),2}\right)^{1-\theta}\left(\dot{f}^0_{1,2}\right)^{\theta} = \left[\dot{f}^0_{p(\cdot),2}, \dot{f}^0_{1,2}\right]_\theta.$$

Applying Theorems 3.1, 2.1 and (1.1) we can write $(L^{q'(\cdot)}(\mathbb{R}^n))' = [L^{p(\cdot)}(\mathbb{R}^n), H^1(\mathbb{R}^n)]_\theta$ where $1/q'(\cdot) = (1-\theta)/p'(\cdot)$ and consequently $L^{q(\cdot)}(\mathbb{R}^n) = [L^{p(\cdot)}(\mathbb{R}^n), H^1(\mathbb{R}^n)]_\theta$ where $1/q(\cdot) = \theta + (1-\theta)/p(\cdot)$.

Let $f \in L^{p(\cdot)}(\mathbb{R}^n) \cap BMO(\mathbb{R}^n)$ and $s = \{(S_\varphi)_Q\}_{Q\in\mathcal{D}}$. From (3.2) and (3.4) we obtain

$$\|s\|_{\dot{f}^0_{q(\cdot),2}} = \left\|s^{1-\theta}s^{\theta}\right\|_{\dot{f}^0_{q(\cdot),2}} \leqslant C\|s\|^{1-\theta}_{\dot{f}^0_{p(\cdot),2}}\|s\|^{\theta}_{\dot{f}^0_{\infty,2}}.$$

This completes the proof of (1.3). $\square$

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